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Symmetric Group of Some Strongly Regular Graphs

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Abstract— Many graph theory problems could be solved by using the principles of group theory, such as symmetric groups, automorphisms of groups and connecting these ideas to the automorphisms of a graph. These algebraic principles play the basic role and could be used as tools to prove tasks related to graph theory.

One area in which Algebra would be adopted and used in the other areas of graph theory is the Strongly Regular Graphs.

Many basic concepts are assumed be well known to the readers of this paper. However, we shall demonstrate all the necessary requirements in order to proceed to the higher levels of our research.

Index Terms— Strongly Regular Graphs, valiancy of a vertex, Adjacency Matrix, Complement of a graph, Peterson Graph, Clebsch Graph, Symmetric Graph, Permutation, Transitivity, Vector Space, Homomorphism, Isomorphism.

1 INTRODUCTION

THE basic definitions of both the group theory and graph theory are introduced prior to the start of the main topics. The first two sections consist of all the basic necessary requirements which would be the background knowledge needed in the next sections. This section also introduces many examples to explain the basic properties.

Section three focuses on the applications of groups on graphs. This includes Automorphisms of graphs and the symmetric groups.

The main results are given in section four. This section mainly deals with the algebra of graphs. Particular attention has been given to the group automorphisms of Clebsch Graph to prove its main structure and properties. These basics are covered in [1] and [6]

1. Basic Properties

1.1 Definition: A graph G consists of two sets $V(G)$ and $E(G)$ called the set of vertices and the set of edges respectively of the graph G togetherwith two functions $i: E \longrightarrow V, t: E \longrightarrow V$.

We say that the edge "e" joins the vertex $i(e)$ to $t(e)$. The vertex $i(e)$ is called the initial vertex of e and $t(e)$ is called the terminal vertex of e. For each element e in E there is an element $\bar{e} \neq e$ in E called the inverse of e, such that $i(\bar{e}) = t(e), t(\bar{e}) = i(e)$ and $\bar{\bar{e}} = e$.

We also say that u and v are adjacent to each other if uv is an edge in $E(G)$,moreover $u \sim v$ if uv is an edge.

1.2 Definition: Let $G = (V, E)$ be a graph. If $v \in V$ then $N_G(v) = \{u: vu \in E\}$ is the set of neighbours of v.

$d(v) = |N_G(v)|$ is the degree of v.

1.3 Definition: If $d(v) = k$, when k is an integer for all $v \in V$ then G is called regular of degree k. So the graph is regular or k-regular, if all its vertices have the same degree k.

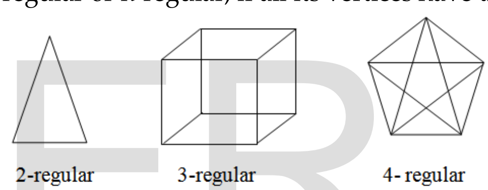

Figure (1)

The following theorem is very well known and given in many other references.

1.4 Theorem (Handshake): Let $G = (V, E)$ be a graph. Then $\sum_{v \in V} d(v) = 2E$

Proof:

Let $X = \{(v, e): v \text{ is an end vertex of } e\}$

Count X in two directions. When we start with e, we will get $|X| = 2|E|$.

Starting with v we get $|X| = \sum_{v \in V} d(v)$.

1.5 Definition: Let$G = (V, E)$ and $G' = G(V', E')$ be graphs. Then a bijective map $\emptyset: V \rightarrow V'$ is an isomorphism provided that $\{u, v\} \in E$ if and only if $(\emptyset(u), \emptyset(v)) \in E'$. $\emptyset$ is called an Automorphism if G=G'.

If $G = (V, E)$ is a graph, $V' \subseteq V$ and $E' \subseteq E \cap [V']^2$ then $G' = (V', E')$ is a sub graph of G expressed as $G' \preccurlyeq G$. If $V' = V$ then G′ is a spanning sub-graph. So all the vertices of V appear but only some of the edges.

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1.6 Definition: Let $V = \{v_0, v_1, \dots, v_k\}$ be a set of pair wis 'hen $P =$

1.7 Definition: Let $P = P(v_0, v_k)$ be as in definition 1.6. Then $C = C^{k+1} = P + v_0 v_k$ is a cycle of length $(k+1)$. The length of a cycle is the number of vertices or edges on it.

1.8 Definition: Let $G = (V, E)$ be graph. If v, v' are vertices, then the distance $d_G(v, \acute{v}) = d(v, v')$ from v to v' is the least k so that there is a path $P(v, v') = P^k \subseteq G$.

The diameter diam (G) of G is the largest k such that $d(v, v') = k$ for some pair of vertices.
The girth $g(G)$ of G is the least k such that $C^k \subseteq G$ for some cycle C^k.

1.9 Definition: A graph is called a tree if it contains no cycle.

If $G = (V, E)$ is a tree then it is connected and $|V| = |E| + 1$. Every connected graph $G = (V, E)$ contains a tree $T = (V, E') \subset G$. This tree is a spanning tree of G.

2- Strongly Regular Graphs

Let G (E, V) be a non-empty graph. The set of neighbours of a vertex v in G is denoted by N_G (v) or N (v). The degree d (v) of a vertex in a graph G (V, E), is the number ∣ E (v) ∣ of edges at v.

d (v) = number of neighbours of v.

2.1 Proposition: The number of vertices of odd degree in a graph is even.

Proof: Suppose the graph G = (V, E) be a graph then G has ½ ∑ d (v) edges so ∑ d (v) in even. The length of a cycle is its number of edges. C^n is referred to a cycle of length n, ie n-cycle.

2.2 Definition: A graph G (V, E) is called complete if every pair of vertices in E are adjacent. This is noted by K^n.

2.3 Definition: A graph G (V, E) in called null if it has no edges at all.

2.4 Definition: The complement of the graph G (V, E) is the graph G whose edge set is the complement of the edge set of G.
ie If G = (V, E), then G′= (V, E′ / E) where (V, E′) is the complete graph

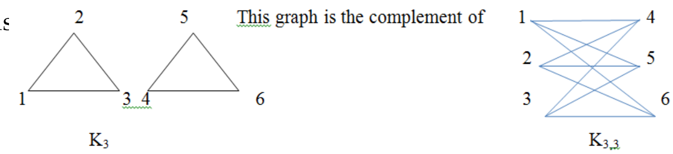

Figure (2)

2.5 Example: let V={1, 2, 3, 4}.Then all the edges of a complete graph will be as follows {12, 13, 14, 23, 24, 34}, these are all 2-element subsets.

Take E={12, 23, 34, 24, 41, 24} to get G = (V, E) which has order ∣ v∣ = 4, as in the following diagram

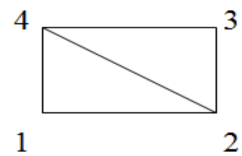

Figure (3)

The following definition is mentioned in van Lint and Wilson [6].

2.6 Definition: A Strongly Regular Graph (v, k, λ, μ) is a graph with v vertices that is regular of degree k and that has the following properties.

1. For any two adjacent vertices x, y, there are exactly λ vertices adjacent to x and to y.
2. For any two non-adjacent vertices x, y there are exactly μ vertices adjacent to x and to y.

Pentagon is Strongly Regular Graph (5, 2, 0, 1).

2.7 Example: Consider the Clebesch graph. The graph has vertices which are all subsets of even cardinality of the set {1, 2, 3, 4, 5}. The neighbouring vertices are those which have difference equals 4. This will form Strongly Regular Graph (16, 5, 0, 2).

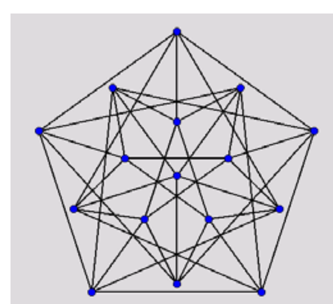

Figure (4)

K_n is strongly regular for any value of n, where as $K_{n,m}$ is strongly regular if and only if n = m.

3. Some Basic Group Properties

3.1 Definition A set G with an operation $*$ is called **group** if the following are satisfied.

1) For all $a, b \in G$, $a * b \in G$.
2) For all $a, b, c \in G$, then $a * (b * c) = (a * b) * c$.
3) There exists an element e , called identity so that $a * e = e * a = a$ for all $a \in G$.
4) For each $a \in G$, exists an element $b \in G$ such that $a * b = b * a = e$, called the inverse of a written by a^{-1} .

A group G is called finite if it contains finite number of elements, and called cyclic if:

$G = \{a^n : a \in G \text{ and } n \in \mathbb{Z}\}$ .A typical examples of finite cyclic group would be $\mathbb{Z}_n$.
Many groups are closely connected to geometrical shapes such as polygons of n sides, including triangle.

Symmetric groups

3.2 Definition A permutation of a set S is a one-to-one, onto mapping from S to itself.

3.3 Definition Let S = $\{1,2,3\}$, the permutations of are:

e = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ a = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$ b = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

c = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ d = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and f = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

The set $\{e, a, b, c, d, f\}$forms a group under composition called the symmetric group S_3 . S_n contains $n!$ elements.

The elements of S_3 can be expressed by :
e = (1)(2)(3), a = (1)(23), b = (2)(13), c = (12)(3), d = (123), f = (132), where $a^2 = b^2 = c^2 = 1$ and $d^3 = f^3 = 1 a = a^{-1}, b = b^{-1}, c^{-1} = c$, $d^{-1} = (3\ 2\ 1) = f$ and $f^{-1} = (2\ 3\ 1) = d$

A permutation $\pi \in S_n$ which interchanges two elements and fixed all the others is called a **transposition.**

Another example is D_n which in he set symmetries of a polygon of n sides. This is called the **Dihedral** group of on n elements.

3.4 Definition A Permutation π in S_n is called a cycle if it has at most one orbit containing more than one element. The **length of a cycle** is determined by the length of its largest orbit.

Eg: $\pi = (1)(2,3)(4)$ is a cycle of length 2.

Let π = $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 1 & 3 & 6 \end{bmatrix}$

Then π is split into the following orbits:
{ 1, 2, 4 }, { 3, 5 }, { 6 }

A permutation of a finite set is even or odd according to whether it can be expressed as a product of an even number of transpositions or the product of an odd number of transpositions.

3.5 Example
$(1, 4, 5, 6)(2, 1, 5) = (1, 6)(1, 5)(1, 4)(2, 5)(2, 1)$
So this is an odd permutation.

3.6 Definition The direct product of the groups $G_1, \dots, G_n$ when n is finite is defined by:

$G_1 * \dots * G_n$ consisting of elements of the form $(g_1, \dots, g_n)$ where $g_i \in G_i$, $1 \leq i \leq n$.

3.7 Definition A homomorphism from a group G to a group $\acute{G}$ is a mapping $f: G \to \acute{G}$ where $f(g_1 g_2) = f(g_1) f(g_2)$ where $g_1, g_2 \in G$.

3.8 Definition An isomorphism is a one-to-one onto homomorphism.

3.9 Definition An automorphism of a group G is an isomorphism of a group G onto itself.
The set of all automorphisms of a group G is denoted by $\boldsymbol{Aut(G)}$.

Theorem 3.1 Let G be a group and $Aut(G)$ be the set of all automorphisms of G.Then $Aut(G)$ forms a group under the compositions of functions.

Proof See [5]

3.10 Definition: Field is a set F together with two binary operations "+" addition and (.) multiplication, satisfying the following axioms:

1 Closure of addition. For every x, y ∈F, we have a + y∈F
2. Commutative of addition. For every x, y∈ F we have x + y = y + x
3. Associative of addition. For x, y, z ∈ F ⇒ x + (y + z) = (x + y) + z
4. The identity of addition. ∃0 ∈ F such that 0 + x = x + 0 = x = x
5 . The inverse of addition. For each x ∈F,∃ (-x) ∈ F such that x + (-x) = (-x) + x = 0
6. Closure of multiplication, x, y ∈F ⇒ x. y ∈ F
7. Commutative of multiplication. x, y ∈F ⇒ x. y = y. x
8. Associative of multiplication. x, y z ∈ F ⇒ x. (y. z) = (x. y). z

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9. Multiplication ideality. $\exists$ 1$\in$ F such that x. 1 = 1. X = x $\forall$x $\in$ F
10. Inverse of multiplication. $\forall$ X $\in$F ,$\exists$$x^{-1}$$\in$ F such that x .x^{-1}=x^{-1}. x = 1
11. Distribution. x. (y + z) = x. y + x. z $\forall$ x, y, z $\in$ F

So (F,+,·) is called a field.

3.11 Example of fields The set of real numbers (R,+,), The set of complex numbers. Z_n = { 0, 1 ,2,….,n}

3.12 Definition: A vector space V is a set with two binary operations of addition (+) and multiplication (·) by scalars, if the following are satisfied.

1. u and v $\in$L $\Rightarrow$u + v $\in$ L (closed under +)
2. u + v = v + u $\forall$ u, v $\in$L (commutative)
3. u+(v + w) = (u + v) + w $\forall$ u , v ,w $\in$L (association)
4. $\exists$ 0 $\in$L, such that 0 + u = u + 0 = u $\forall$ u $\in$L (Identity)
5. For each u $\in$L, $\exists$ - u $\in$L such that: u + (-u) = (-u) + u = o (addition inverse)
6. For any real number k, k u $\in$L, where u $\in$L
7. k (u + v) = k u + k v ,k is scalar
8. (k + l) u = k u + l u , k and ℓ are real
9. K (l u) = k l u ,k and l are real and u $\in$L
10. L .u = u $\forall$ u $\in$ L

4. Groups of Graphs

4.1 Definition Let $G = (V, E)$ be a finite graph. An automorphism of G is a permutation of the vertex set that satisfies the condition $\{u_i, u_j\} \in E(G)$ if and only if $\{\emptyset(u_i), \emptyset(u_j)\} \in E(G)$.

The Automorphism Group of G is the set of permutations of the vertex set that preserve adjacency. See [3]
$Aut(G) = \{\pi \in Sym(v) : \pi(E) = E\}$.

4.1 Theorem The set $Aut(G)$ of all group automorphisms of a group G forms a group under compositions of functions

Proof See [3]

4.2 Definition An edge automorphism on a graph $G = (V, E)$ is a permutation π on the set of edges of $G(E)$ satisfying e_i, e_j are adjacent if and only if $\emptyset(e_i), \emptyset(e_j)$ are also adjacent.

4.2 Theorem $Aut(K_n) \simeq S_n$ (isomorphic).

Proof K_n contains n vertices which are all connected to each other. Each vertex is connected to $n-1$edges . Each vertex from K_n is mapping to another vertex. The other $n-1$ vertices are connected to $n-2$ vertices and so on .
Therefore, $Aut(K_n)$ contains $n(n-1)(n-2) \ldots 2 \times 1$ elements and this given by $n!$.
Since S_n contains $n!$ Elements, so each element in $Aut(K_n)$ is mapped to an element in S_n. Therefore, $Aut(K_n) \simeq S_n$

4.3. Automorphisms of Peterson Graph

Peterson Graph

To construct the Peterson graph, draw a cycle of length 5. At each vertex draw one additional to a new vertex. Draw two new edges from the new vertex producing a star in the centre. The graph is called Peterson graph which is 3-regular graph.

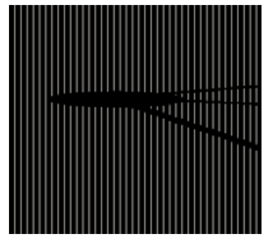


Figure 6

Another way of constructing Peterson graph is given in [4]. It is also shown that, Peterson graph contains no 3-cycles nor 4-cycle.

4.1 Lemma The symmetric group S_5 acts as a group of automorphism of P.

Proof: See [4]

The following theorem was concluded in [4]

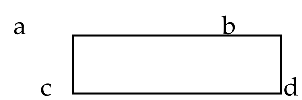
4.3 Theorem The automorphism group of Peterson graph is isomorphic to the symmetric group S_5.

Algebra of Graphs

4.4 Definition The adjacency matrix M for a graph G is a matrix which has raws and columns are associated with the vertices of the graph, such that each entry a_{ij}=1 if i and j are adjacent vertices and a_{ij} = 0 otherwise.

Example Let G be the graph given by

Figure 7



The adjacency matrix of G will be as follows:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

M is symmetric with zeros on the diagonal.

4.2 Lemma If G (V, E) is a Strongly Regular Graph and G = (n, k, λ, μ), then k (k – 1 – λ) = μ (n – k -1)

Proof Let $v \in V$,
$N = N_G(v), N' = V \setminus \{v\} \setminus N$
Let $x \in N$, then the number of edges from x to a vertex in N is λ.
So there are k -1 – λ edges from x to N'
Therefore, the number of edges from N to N' is (k –1 – λ)

Now let $y \in N'$. Then there will be μ edges at y to N, as vy is not an edge
There are μ edges from N' into N. So $|N'|$ = n – k -1
It is also well known that the ij edge of M^2 is the number of walks from v_i to v_j of length 2.

We now prove the following relationship which is stated in [5]

4.4 Theorem If G is a Strongly Regular Graph and G = (n, k, λ, μ) and M is its adjacency matrix, then $M^2 = \lambda M + \mu(J - I - M) + kI$), where J is an $n \times n$ matrix with all entries 1 and I is $n \times n$ identity matrix.

Proof If $a_{ij} = 0$ in M^2, then the ij entry of M^2 gives $i \neq j$

If $a_{ij} = 1$ in M^2, then the ij entry of M^2 gives $i = j$

Hence $M^2 = kI + \mu(J - I - M) + \lambda M$

$$M^2 = \lambda M + \mu(J - I - M) + kI$$

Remarks

The number λ is an **Eigenvalue** of a matrix M if and only if det $(M - \lambda I) = 0$.
For each λ we solve the equation $(M - \lambda I)x = 0$ or M x = λx to find an **Eigenvector** of x.
M^2 has the same Eigenvector of M. Moreover, the Eigenvectors of M^2 are the squares of the Eigenvectors of M.
Let M be an $n \times n$ matrix and λ be the eigenvalues of M, then the set of all eigenvectors corresponding to the eigenvalues λ is called the Eigenspace of λ. This is a subspace of $\mathbb{R}^2$.

Symmetric Groups and Clebsch Graph

Let V be the collection of all subsets from {1, 2, 3, 4, 5} which have even size. Define the set of vertices as follows:
Two vertices v_1 and $v_2 \in V$ will be adjacent if $|v_1 \cup v_2| - |v_1 \cap v_2| = 4$, as v_1 and v_2 are subsets of {1, 2, 3, 4, 5 }.
It is very well known that this graph is Strongly Regular and the number of its vertices v =16. However, we prove that v =16 using Symmetric Group Theory approach. [2]

4.5 Theorem Clebsch Graph is strongly regular with 16 vertices. Find |E|, k, λ and μ.
Determine the structure of the graph formed by the non-neighbours of a given vertex

Proof Represent each set from {1, 2, 3, 4, 5 } by a vector in the field GF (2, 3) - three dimensional vectors of the entries from { 0, 1 }.ie { 1, 2, 5 } becomes {1, 1, 0, 0, 1 }
So the vertices of G are even weight vectors in GF (2, 5).

These are the ones which have even number of entries of 1.
This set of vectors is closed under vector space addition.
Therefore, the vertex set will be a subspace of GF (2, 5) of dimension 1, resulting to the fact that $|V| = 16$.

Using vector addition in GF (2, 5) we find that the vertices v_1 and v_2 will be adjacent if $v_1 + v_2 = e_i$ where e_i = (0,..,1,, 0) is a basis of V and 1 is at the i^{th} position and i = 1, …, 5
Fixing a vertex $v \in V$, the map $x \to x + a$ will form an automorphism. This is because $x \in V$ and $x + y = e_i$ we get $(x + a) + (y + a) = e_i$, whereas $a + a = 2a = 0$ for $a \in GF(2,5)$. Therefore, V operates as a transitive group of automorphism on itself.

That means for any vertices $v_1, v_2 \in V, \exists$ a mapping v_1 to v_2. This shows that the graph is regular. As the vertices can be permuted, this operation will form a group of automorphisms equivalent to S_5.
Applying this procedure on all pairs u, v we get a Strongly Regular Graph condition satisfied.

As the automorphisms are transitive, so we set $x = (0,0,0,0,0)$, while $y = e_i$ if y is a neighbor representing a 2-element subset.

The group S_5 preserves both the neighbours and the non-neighbours of x and acts transitively on each set. So it can be checked that $\lambda = 0$ and $\mu = 2$, and | E | = 40 edges, by using the Handshake Lemma.

Now we look at the non-neighbours of $x = (0,0,0,0,0)$.
There are $\binom{5}{2}$ vectors of the form $y = (1,1,0,0,0)$.
However these are the vertices of Peterson graph.

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